Lefschetz-thimble path integral and SSB

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Introduction

Motivation

Path integral with **complex weights** appear in many important cases:

- Finite-density lattice QCD,
  spin-imbalanced nonrelativistic fermions
- Gauge theories with topological θ terms
- Real-time quantum mechanics

The complex phase **obstructs** our naive **nonperturbative methods**, such as saddle point methods, Monte Carlo integration, etc.

Example: Airy integral

Let's consider a one-dimensional oscillatory integration:

$$\mathrm{Ai}(a) = \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi} \exp \mathrm{i} \left(\frac{x^3}{3} + ax \right).$$

RHS is well defined **only if** $\mathrm{Im} a = 0$, though $\mathrm{Ai}(z)$ is **holomorphic**.

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Let us regard x as a **complex** variable. By deforming $\mathbb{R}$ so that

$$\mathrm{Re}\left[\mathrm{i}S(x,a)\right] \to -\infty \qquad \text{as} \qquad |x| \to \infty,$$

we can allow a being complex from the first!.

Steepest descent method

What is the most appropriate contour for our purpose?

$\mathrm{Re}[\mathrm{i}S(x,a)]$ should be made as small as possible.
$\Leftrightarrow$ The contour should be **perpendicular** to $\mathrm{Re}[\mathrm{i}S(x,a)] = \mathrm{cosnt}.$

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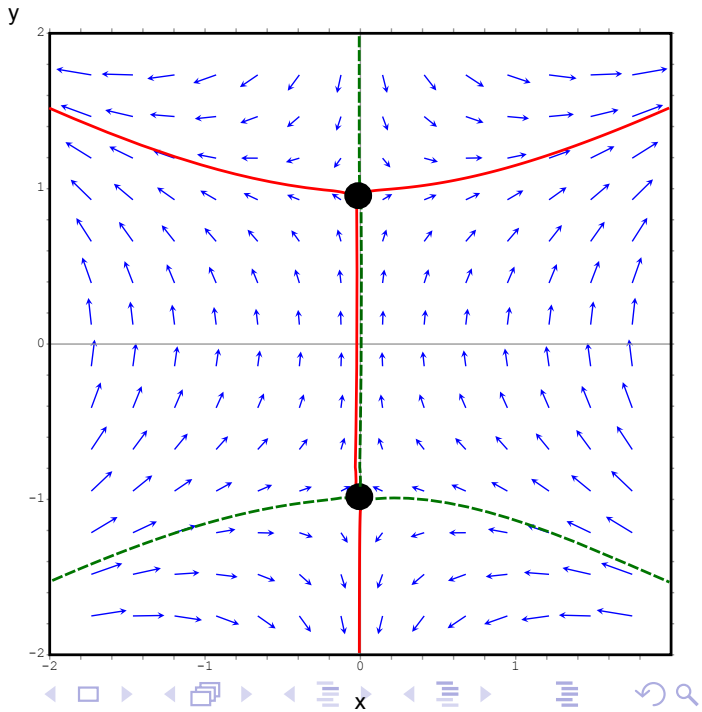
Holomorphy says that the path satisfying

$$\mathrm{Im}[\mathrm{i}S(x,a)] = \mathrm{const.}$$

is the **steepest descent** one.



Multiple integral: Lefschetz-thimble method

Oscillatory integrals with **many variables** can be evaluated using the "'steepest descent" cycles $\mathcal{J}_\sigma$:

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \; e^{\mathrm{i}S(x)} = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathrm{d}^n z \; e^{\mathrm{i}S(z)}.$$

$\mathcal{J}_\sigma$ are called Lefschetz thimbles, and $\mathrm{Im}[\mathrm{i}S]$ is constant on it.

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New challenges for unveiling nonperturbative natures of QFTs:

- Analytic properties of partition function for CS theories [Witten]
- Sign problem of the lattice simulation [AuroraScience, ...]
- Geometrization of resurgence transseries [Basar, Dunne, Unsal; Cherman, Dorigoni, Unsal]
- Real-time description of tunneling [YT, Koike; Cherman, Unsal]

Motivation
What is SSB in this context?

In this talk, we consider two questions using zero-dimensional fermionic models with N species:

- What happens when discrete symmetry is broken at large N?
- Can we also treat continuous symmetry?

Phase transition associated with discrete symmetry: Gross–Neveu-like model

It will appear on today's arXiv...

0-dim. Gross–Neveu-like model

The partition function of our model study is the following:

$$Z_N(G,m) = \int \mathrm{d}\bar{\psi}\mathrm{d}\psi \ \exp\left\{\sum_{a=1}^{N} \bar{\psi}_a(i\not{\partial} + m)\psi_a + \frac{G}{4N}\left(\sum_{a=1}^{N}\bar{\psi}_a\psi_a\right)^2\right\}.$$

The Hubbard–Stratonovich transformation gives

$$Z_N(G,m) = \sqrt{\frac{N}{\pi G}}\int_{\mathbb{R}} \mathrm{d}\sigma\, \mathrm{e}^{-NS(\sigma)},$$

with

$$S(\sigma) \equiv \frac{\sigma^2}{G} - \log[p^2 + (\sigma+m)^2].$$

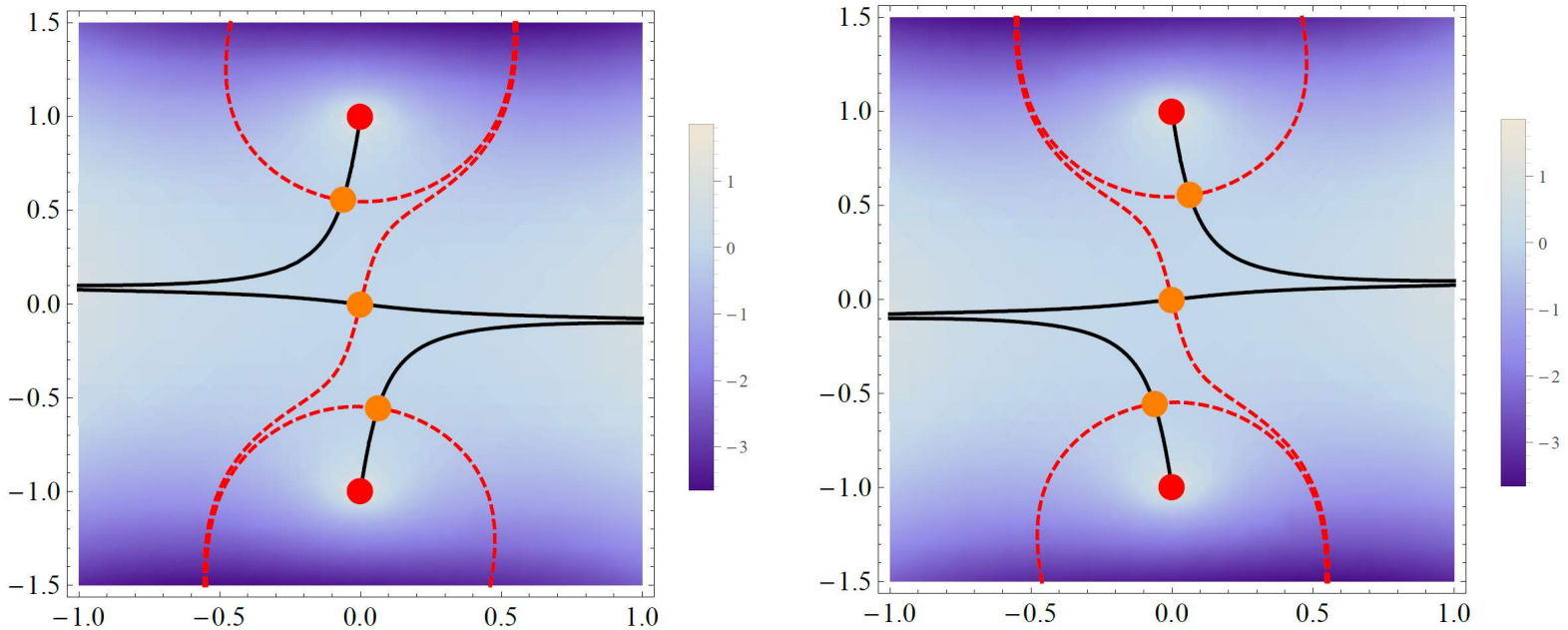
For simplicity, we put $m = 0$ in the following.

Properties of $S(\sigma)$

S has three saddle points:

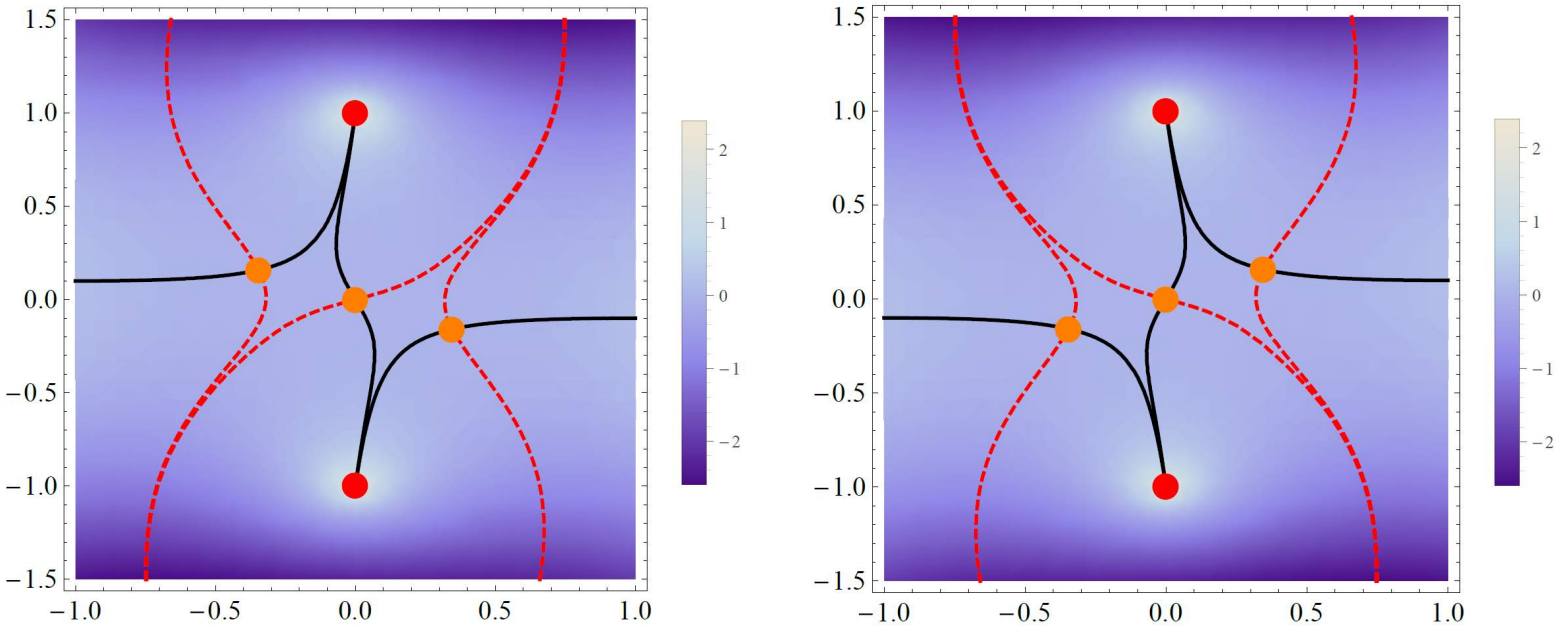
$$0 = \frac{\partial S(z)}{\partial z} = \frac{2z}{G} - \frac{2z}{p^2 + z^2} \quad \Longrightarrow \quad z = 0,\ \pm\sqrt{G - p^2}\,.$$

Figures for the case $G = 0.7e^{\mp 0.1\mathrm{i}}$ and $p^2 = 1$:



Properties of $S(\sigma)$

Figures for the case $G = 1.1e^{\mp 0.1\mathrm{i}}$ and $p^2 = 1$:



From these figures, we learn that, for real G,

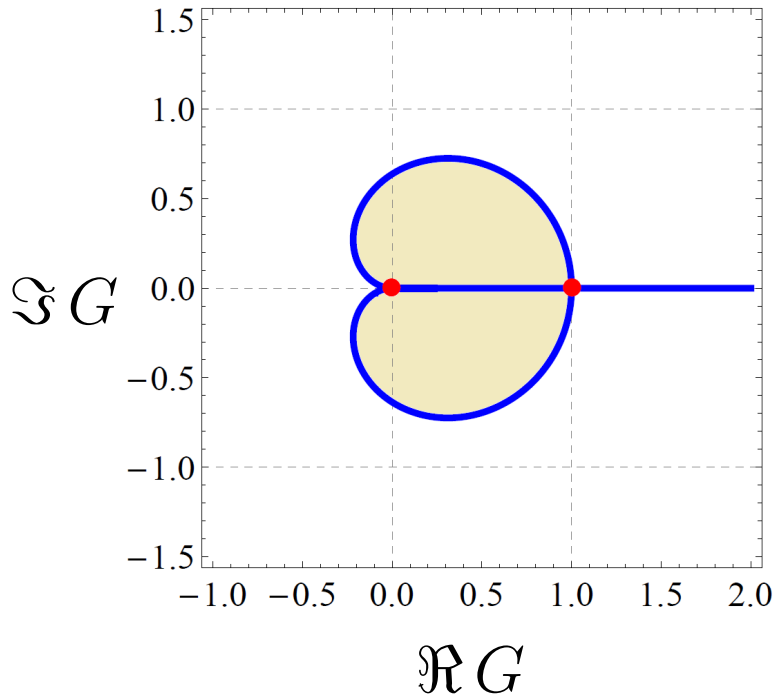
- $z = 0$ is the unique critical point contributing to Z if $G < p^2$.
- All three critical points contribute to Z if $G > p^2$.

Stokes phenomenon

The difference of the way of contribution can be described by **Stokes phenomenon**.

$\Leftarrow$ At some **special values** of coupling, several critical points are connected by the flow. [Witten, 2010]

Figures of G-plane for $\mathrm{Im}S(0) = \mathrm{Im}S(z_\pm)$:



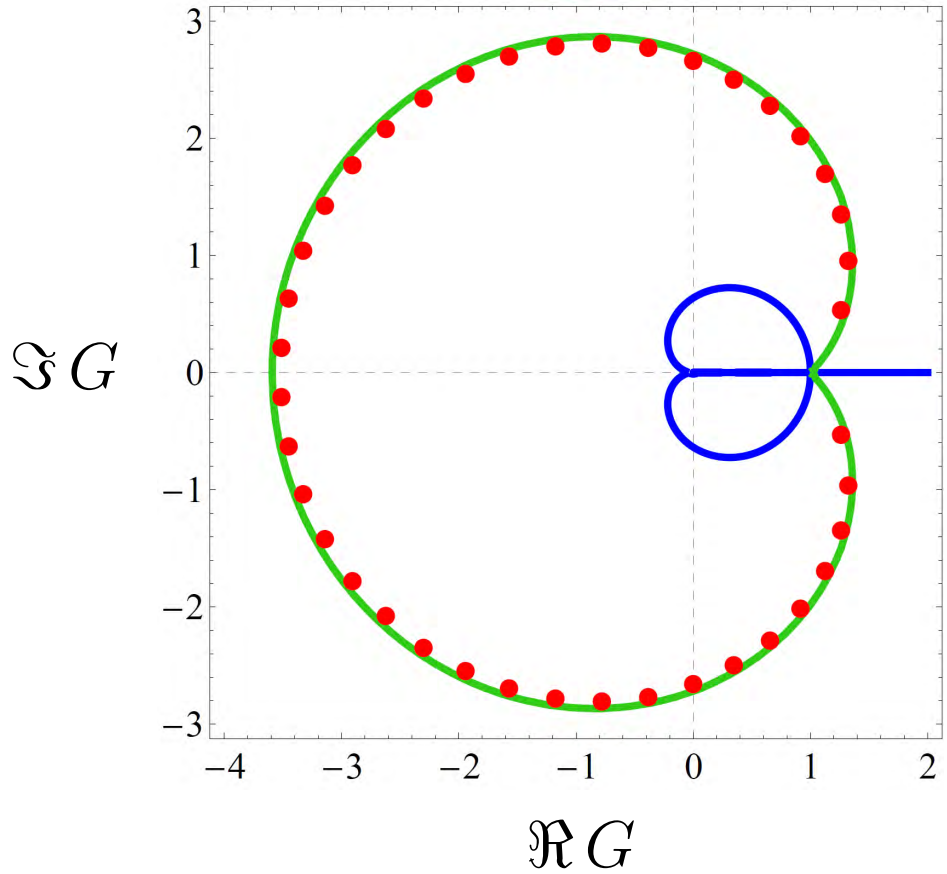
Dominance of contribution

The Stokes phenomenon tells us the **number** of Lefschetz thimbles contributing to Z_N.
However, it does **not** tell which thimbles give main contribution.

In order to obtain $\langle\sigma\rangle \neq 0$ in the large-N limit, $z_\pm$ should dominate $z = 0$.

$$\Rightarrow \qquad \mathrm{Re}S(z_\pm) \leq \mathrm{Re}S(0)$$

Connection with Lee–Yang zero



Blue line: $\mathrm{Im}S(z_\pm) = \mathrm{Im}S(0)$.
Green line: $\mathrm{Re}S(z_\pm) = \mathrm{Re}S(0)$.
Red points: Lee-Yang zeros at $N = 40$. [Kanazawa, YT]

Conclusions for studies with GN-like models

1. Decomposition of the integration path in terms of Lefschetz thimbles is useful to visualize different phases.
2. The possible link between thimble decomposition and Lee–Yang zeros is indicated.

Lefschetz thimbles with continuous symmetry

arXiv:1412:1891, & the new article appearing on today's arXiv

Short note on technical aspects

In this section, we consider $O(2)$-symmetric linear sigma models.
$\Rightarrow$ Multiple integration necessarily appears.

Short review on Lefschetz thimbles:
Model integral:

$$Z = \int_{\mathbb{R}^2} \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \exp S(\sigma_1, \sigma_2).$$

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What properties are required for Lefschetz thimbles $\mathcal{J}$?

1. $\mathcal{J}$ should be a two-dimensional object in $\mathbb{C}^2$.
2. $\mathrm{Im}S$ should be constant on $\mathcal{J}$.

Short note on technical aspects

Complexified variables $(a = 1, 2)$: $\xi_a = \sigma_a + \mathrm{i}\eta_a$.
Regard σ_a as **coordinates** and η_a as **momenta**, so that **Poisson bracket** is given by

$$\{f, g\} = \sum_{a=1,2} \left[\frac{\partial f}{\partial \sigma_a} \frac{\partial g}{\partial \eta_a} - \frac{\partial g}{\partial \sigma_a} \frac{\partial f}{\partial \eta_a} \right].$$

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Hamilton equation with the Hamiltonian $H = \mathrm{Im}S[\xi]$:

$$\frac{\mathrm{d}f(x, \eta)}{\mathrm{d}t} = \{H, f\}$$

This is **Morse's flow equation**: $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}S \leq 0$.
$\Rightarrow$ We can find two good directions for $\mathcal{J}$ around saddle points!
[Witten, 2010]

Lefschetz thimble with approximate $O(2)$ symmetry

Let's consider approximately $O(2)$-symmetric sigma model:

$$S(\boldsymbol{\sigma}) = \frac{1}{4}(\boldsymbol{\sigma}^2 - 1)^2 + \varepsilon\sigma_1.$$

"Angular momentum" gives an **extra** conserved quantity at $\varepsilon = 0$!

$$p_\theta = \eta_2\sigma_1 - \eta_1\sigma_2.$$

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At $\varepsilon = 0$, the flow is **singular**, and naive construction of $\mathcal{J}$ fails down.
Under this circumstance, how does the flow look like?

Structure of the flow equation

Hamilton equation (= flow equation):

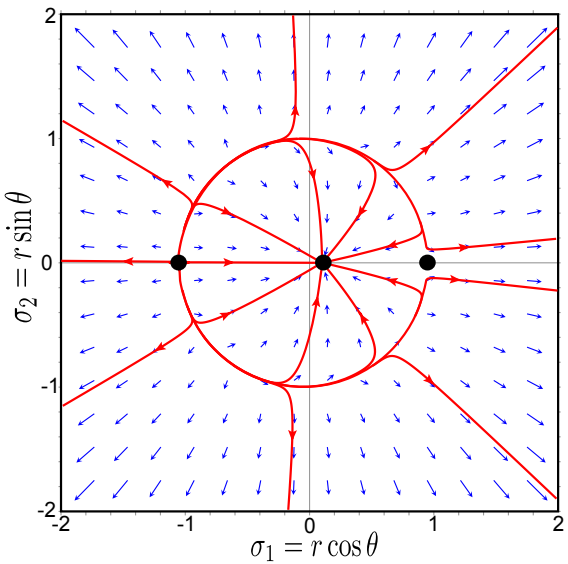
$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= r\left(r^2 - \frac{p_\theta^2}{r^2} - 3p_r^2 - 1\right) + \varepsilon\cos\theta, \\
\frac{\mathrm{d}p_r}{\mathrm{d}t} &= p_r\left(1 - 3r^2 + p_r - \frac{p_\theta^2}{r^2}\right) - \varepsilon\frac{p_\theta}{r^2}\sin\theta, \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\frac{2p_r p_\theta}{r} - \varepsilon\frac{\sin\theta}{r}, \\
\frac{\mathrm{d}p_\theta}{\mathrm{d}t} &= \varepsilon\left(p_r\sin\theta + \frac{p_\theta}{r}\cos\theta\right).
\end{aligned}$$

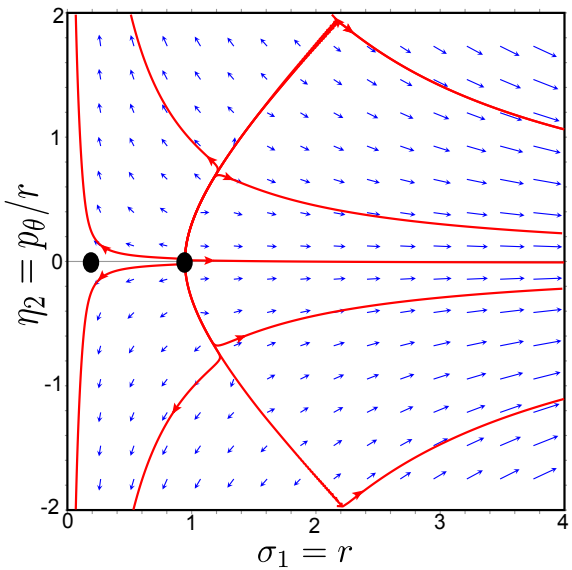
In the limit $\varepsilon \to 0$, flows along symmetric direction disappears.
$\Rightarrow$ Naive construction of Lefschetz thimbles breaks down at $\varepsilon = 0$.

Flow with approximate $O(2)$ symmetry

Set of saddle points at $\varepsilon = 0$: $\boldsymbol{\sigma}^2 - \boldsymbol{\eta}^2 = 1$ & $\boldsymbol{\sigma} \cdot \boldsymbol{\eta} = 0$.

Flows in the $\sigma_1\sigma_2$-plane & in the $\sigma_1\eta_2$-plane for small ε:



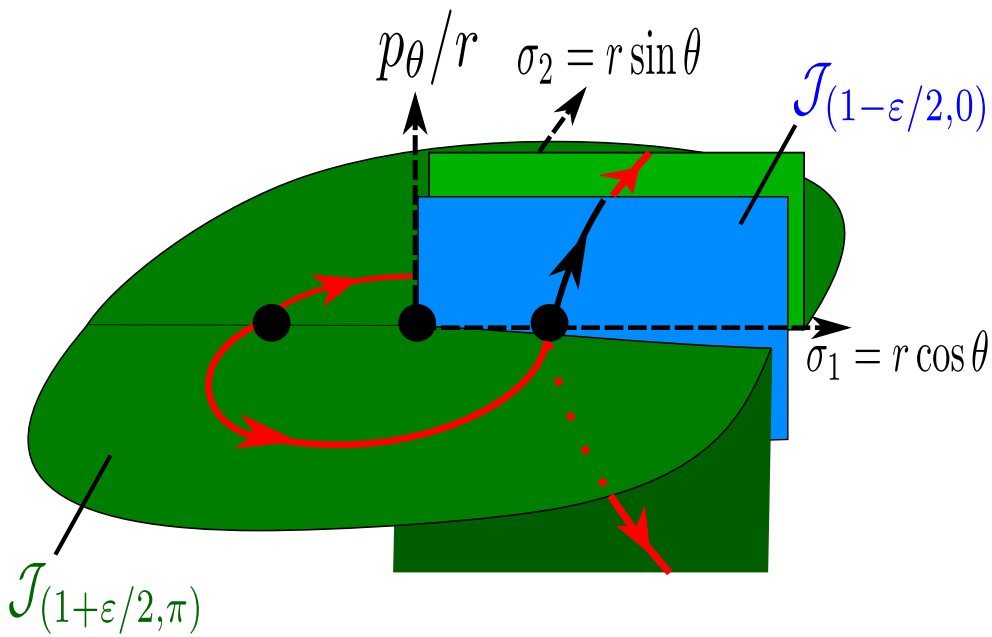


- Due to the approximate $O(2)$ symmetry, flow walks around the pseudo saddle points. [YT, arXiv:1412:1891]

Flow with approximate $O(2)$ symmetry

Schematic pictures for Lefschetz thimbles of the system

$$S = e^{\mathrm{i}0^+} \left(\frac{1}{4}(\boldsymbol{\sigma}^2 - 1)^2 + \varepsilon\sigma_1 \right).$$



$\Rightarrow$ In order to get an $O(2)$ symmetric result, we **must sum up** two thimbles $\mathcal{J}_{(1+\varepsilon/2,\pi)}$ and $\mathcal{J}_{(1-\varepsilon/2,0)}$. [YT, arXiv:1412:1891]

Singularity of symmetry-restoration limit

What is the fate of $\mathcal{J}_{(1+\varepsilon/2,\pi)}$ and $\mathcal{J}_{(1-\varepsilon/2,0)}$ in the limit $\varepsilon \to 0$?

Integration on $\mathcal{J}_{(1-\varepsilon/2,0)}$:

$$\int_{\mathcal{J}_{(1-\varepsilon/2,0)}} \mathrm{d}^2 z \exp\left[-S/\hbar\right]$$

$$\sim \quad -\mathrm{i}\int_0^\infty \lambda \mathrm{d}\lambda \exp\left(-\frac{(\lambda^2-1)^2}{4\hbar}\right) \underline{\int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-\frac{\varepsilon\lambda}{\hbar}\cosh\phi\right)}.$$

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Logarithmic divergence: $\int \mathrm{d}\phi \exp\left(-\frac{\varepsilon\lambda}{\hbar}\cosh\phi\right) \sim 2\ln\frac{\hbar}{\varepsilon}$.

$\Rightarrow$ Thimbles related by the symmetry need to be summed up to give finite results!

Nambu–Jona-Lasinio-like model

Let's apply this observation to the following NJL-like model:

$$Z_{\mathrm{U}(1)} = \int \mathrm{d}\bar{\psi}\mathrm{d}\psi \exp\left(\sum_{a=1}^{N} \bar{\psi}_a (i\not{p} + m)\psi_a + \frac{G}{4N}\left\{\left(\sum_{a=1}^{N}\bar{\psi}_a\psi_a\right)^2 + \left(\sum_{a=1}^{N}\bar{\psi}_a i\gamma_5\psi_a\right)^2\right\}\right) .$$

Bosonization:

$$Z_{\mathrm{U}(1)} = \frac{N}{\pi G}\int_{\mathbb{R}^2} \mathrm{d}\sigma\mathrm{d}\pi \exp\left(-NS(\sigma,\pi)\right) ,$$

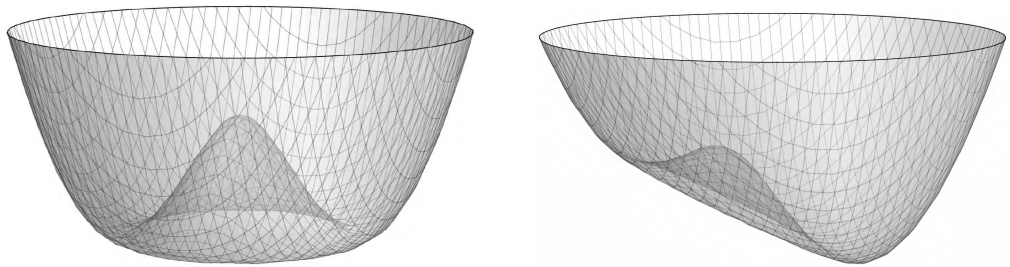
with

$$S(\sigma,\pi) \equiv -\log\left(p^2 + (\sigma+m)^2 + \pi^2\right) + \frac{\sigma^2+\pi^2}{G} .$$

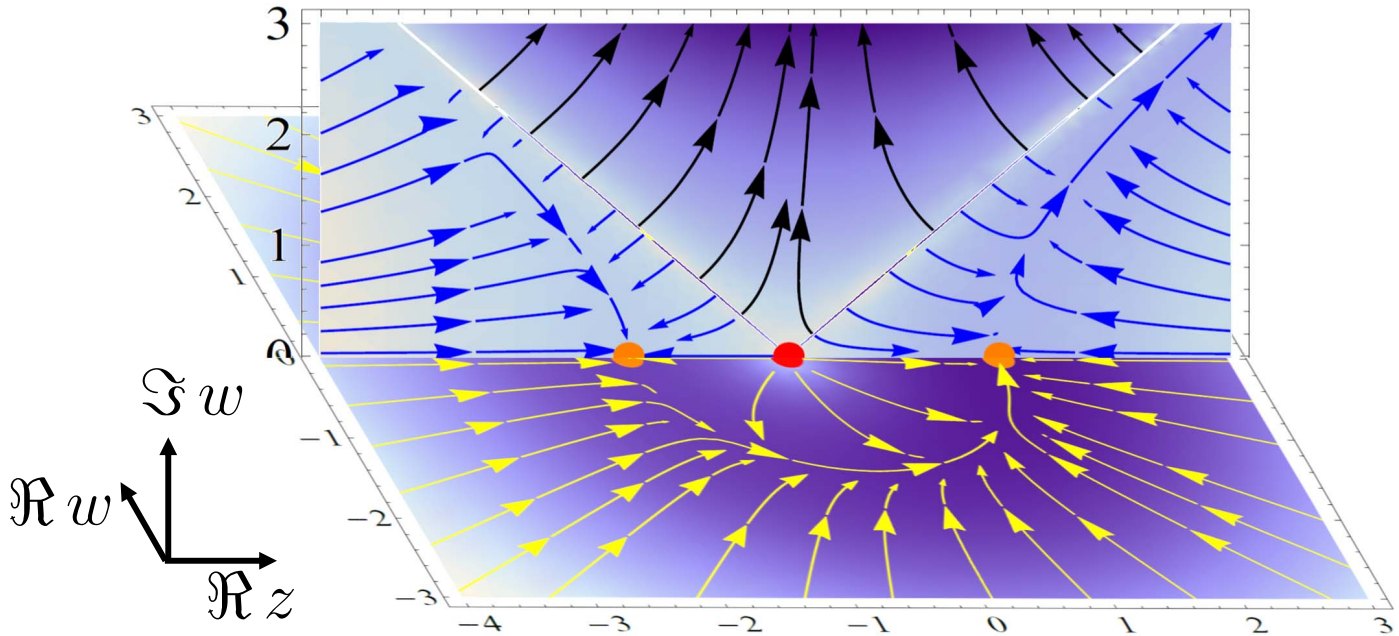
Lefschetz thimbles for NJL-like model

In this case, $m(\neq 0)$ breaks the $O(2)$ symmetry:

$m = 0$ $\quad$ $m = 0.4$



Flow for NJL-like model at finite m (& $p = 0$) [Kanazawa, YT]:



Conclusions for the continuous symmetry

- Lefschetz thimbles with approximate continuous symmetries are computed.
- The remnant of the symmetry implies **existence of slow flows** on the set of pseudo saddle points.
- Such slow flow relates different thimbles via the approximate symmetry.

Summary and Perspective

Summary

- Zero-dimensional integrals are studied via Lefschetz-thimble methods.
- Phase transition under large-N limit is scrutinized, and the connection to Lee–Yang theorem is indicated.
- We discuss how approximate continuous symmetry affects structures of Lefschetz thimbles.

Perspective

- Realistic statistical model should be studied.
  Do these observation for phase transition still hold?